Pulses and waves for a bistable nonlocal reaction–diffusion equation

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Abstract

A bistable nonlocal reaction-diffusion equation is studied. Solutions in the form of simple and periodic travelling waves, single and multiple pulses are observed in numerical simulations. Successive transitions from simple waves to periodic waves and to stable pulses are described.

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1. Introduction

Nonlocal reaction-diffusion equations

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + au^2(1 - J(u)) - \sigma u, \tag{1}$$

where

$$J(u) = \int_{-\infty}^{\infty} \phi(x-y)u(y,t)dy, \qquad \int_{-\infty}^{\infty} \phi(y)dy = 1.$$

describes various biological phenomena such as emergence and evolution of biological species and the process of speciation in a more general context [8], [9]. An important property of such equations is that they have solutions in the form of periodic travelling waves [1], [2], [3]. Such solutions do not exist for the usual (scalar) reaction-diffusion equations. In this work we present a new type of solutions of this equation, single and multiple pulses, and show how they are related to periodic travelling waves. We will consider the kernel $\phi(x)$ in the form of a step-wise constant function:

$$\phi(x) = \begin{cases} 1/(2N) & , \quad |x| < N \\ 0 & , \quad |x| \geq N \end{cases}.$$

In the limit of small N we obtain the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + au^2(1 - u) - \sigma u. \tag{2}$$

It is well known that it can have travelling wave solutions and solutions in the form of stationary pulses, that is positive solutions decaying at infinity. Travelling waves are asymptotically stable with shift while pulses are unstable. Existence of waves for the nonlocal equation (1) is proved for sufficiently small N [1], [2]. Existence of pulses can be proved by a similar method. Moreover, travelling waves are stable if N is small enough and pulses are unstable.

The second limiting case of equation (1) is that of large N. Instead of the kernel $\phi(x)$ we now consider the kernel $\psi(x) = 2N\phi(x)$. The limiting equation becomes

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + au^2(1 - I(u)) - \sigma u, \qquad I(u) = \int_{-\infty}^{\infty} u(y,t)dy. \tag{3}$$

This equation does not have travelling waves but it has stationary pulses. The integral term in this equation can make them stable [8]. Therefore we can expect that stable pulses also exist for equation (2) if N is sufficiently large.

Thus, for sufficiently small N nonlocal equation (1) has stable waves and unstable pulses. For sufficiently large N, it can have stable pulses but there are no waves. In this work we will study transition of solutions of this equation from stable waves to stable pulses as N increases.

2. Travelling waves

If $\sigma/a < 1/4$, then equation (1) has three homogeneous in space stationary solutions, $u_+ = 0$, and two other solutions u_0 and u_-, $u_0 < u_-$, of the equation $u(1-u) = \sigma/a$. The homogeneous in space stationary solution $u = u_-$ of equation (1) can lose its stability resulting in appearance of periodic in space solutions [4], [5], [6]. If we consider a localized in space perturbation of this homogeneous in space solution, then it propagates as a periodic wave. The speed and the amplitude of this wave depends on parameters. In the linear approximation, the speed can be estimated through the maximal positive eigenvalue [10].

Let c_0 be the speed of the wave with the limits $w(\pm\infty) = u_\pm$, which exists at least for sufficiently small N, and c_p be the average speed of the periodic wave which effectuates transition from u_- to the periodic in space stationary solution. If the support N of the kernel $\phi(x)$ is sufficiently small, then we can use the linear approximation to describe the propagation of the perturbation, and its speed converges to zero. Therefore for sufficiently small N, $c_0 < c_p$, and the $[c_+, c_-]$-wave runs away from the periodic wave.

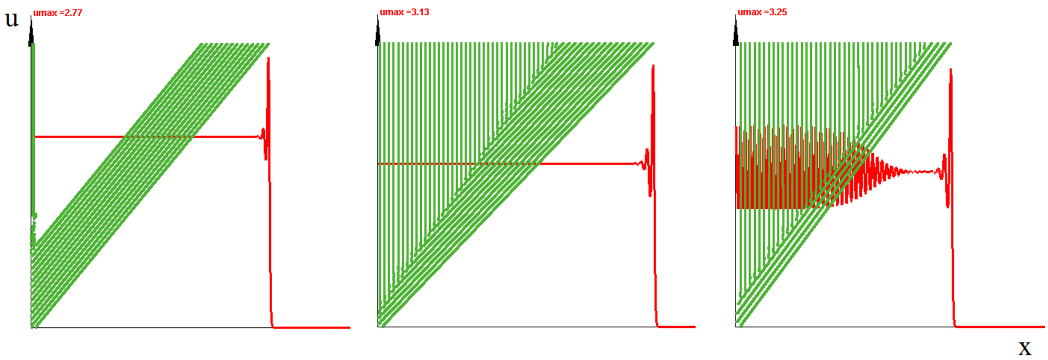

Figure 1: Numerical simulations of equation (1). Snapshot of solution $u(x,t)$ is shown in red. Green lines represent positions of the maxima of solution in the (x,t)-plane. If u_- is stable, then then there is a simple $[u_+, u_-]$-wave ($N = 3$, left). If it is unstable, then the periodic perturbation can propagate slower than this wave ($N = 3.6$, middle) or with the same speed ($N = 3.8$, right).

Figure 1 shows different regimes of wave propagation. If the solution u_- is stable, then it is a simple $[u_+, u_-]$-wave. It is not monotone with respect to x. The green lines in the left figure show the position of the maxima of solution. They move altogether with the wave front. If N is greater than the critical value $N_c \approx 3.6$, then the homogeneous in space stationary solution becomes unstable and a periodic in space stationary solution emerges behind the $[u_+, u_-]$-wave. If N is close to the critical value, then the amplitude and the speed of propagation of the periodic wave are small. It propagates slower than the $[u_+, u_-]$-wave (Figure 1, middle). For a greater N, they propagate with the same speed but the periodic wave stays at some distance behind the $[u_+, u_-]$-wave (Figure 1, right). Its influence is exponentially small, and the $[u_+, u_-]$-wave can still be considered as having a constant speed and profile. Finally for sufficiently large values of N, the two waves merge forming a single periodic wave (Figure 2, middle).

3. Single and mutiple pulses

3.1. Unstable pulses

Consider the equation

$$dw'' + aw^2(1 - w) - \sigma w = 0. \tag{1}$$

If $\int_0^{u_-} F(u)du > 0$, then it has a positive solution $w_0(x)$ which decays at infinity. This solution is unstable since the corresponding linearized operator has a positive eigenvalue [8]. Along with equation (1) we consider the corresponding nonlocal equation

$$dw'' + aw^2(1 - J(w)) - \sigma w = 0, \tag{2}$$

where

$$J(w) = \int_{-\infty}^{\infty} \phi(x - y)w(y)dy, \quad \phi(x) = \begin{cases} 1/(2N) & , \quad |x| < N \\ 0 & , \quad |x| \geq N \end{cases} .$$

It can be proved by the perturbation technique similar to travelling waves [1], [2] that for all N sufficiently small there exists a pulse solution of this equation. The proof uses the implicit function theorem and the spectral properties of the linearized operator. This pulse solution is unstable for sufficiently small N.

3.2. Stable pulses

Next, consider the equation

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + au^2(1 - K(u)) - \sigma u, \qquad K(u) = \int_{-\infty}^{\infty} \psi(x - y)u(y, t)dy, \tag{3}$$

where the kernel of the integral differs from the kernel $\phi(x)$ in the integral $J(u)$ by the factor $2N$. The corresponding stationary equation

$$dw'' + aw^2(1 - K(w)) - \sigma w = 0, \qquad K(w) = \int_{-\infty}^{\infty} \psi(x - y)w(y)dy \tag{4}$$

is close to the equation

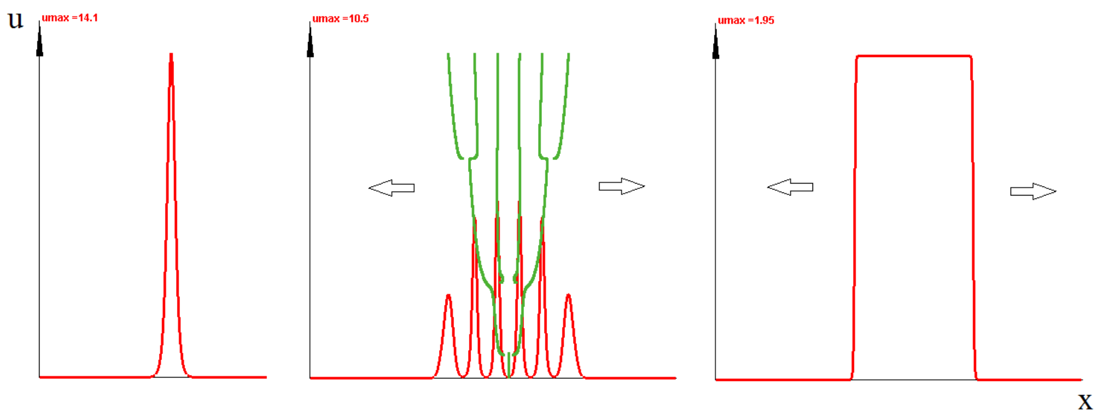

Figure 2: Stable pulse for $N = 20$ (left). The same pulse becomes unstable for $N = 8$, and a periodic travelling wave forms (middle). For $N = 0.5$, it is a usual wave instead of the periodic wave. The values of other parameters: $d = 0.2, a = 1, \sigma = 0.1$.

$$dw'' + aw^2(1 - I(w)) - \sigma w = 0, \qquad I(w) = \int_{-\infty}^{\infty} w(y)dy. \tag{5}$$

if N is sufficiently large. It can be proved that equation (5) has two positive decaying at infinity solutions for any $a > a_0$, where a_0 depends on σ [8]. If $a < a_0$, then such solutions do not exist. It allows one to prove existence of pulses for equation (4) for all N sufficiently large [11]. The proof of existence uses spectral properties of the linearized operator

$$Lu = du'' + 2aw(1 - I(w))u - \sigma u - aw^2 I(u).$$

It is a Fredholm operator in weighted Hölder spaces with the zero index and one-dimensional kernel. It is invertible on subspace of even functions. These properties allow the application of the implicit function theorem to prove of existence of solutions for sufficiently large N.

The spectrum of the operator L determines stability of the pulse solution of equation (5) as a stationary solution of the corresponding evolution equation. Numerical simulations show that this solution is stable. However it is not proved that the spectrum of operator L lies in the left-half plane of the complex plane. This property is verified for some related but different operators [7].

Assuming that all eigenvalues of the operator L lie in the left-half plane except for a simple zero eigenvalue related to translation invariance of stationary solutions, we obtain similar properties of the linearized operator corresponding to equation (4). Therefore we can expect that the pulse solutions of this equation are asymptotically stable with respect to small perturbations. Figure 2 (left) shows a stable pulse for a large value of N. If we decrease N, this pulse becomes unstable and initiates a travelling wave propagating in both directions (Figure 2, middle and right). It can be a simple wave or a periodic wave depending on parameters.

3.3. Multiple pulses

The results of the previous sections show that there are two alternative cases. In the first case the wave is stable and the pulse is unstable. In the second case the pulse is stable, and the wave is not observed in numerical simulations. Most likely it does not exist.

In order to study behavior of solutions in the case of stable pulses in more detail, we vary initial conditions. Depending on the width of the support of the initial condition considered in the form of a step-wise constant function, we obtain single or multiple pulses. If the support of the initial condition is sufficiently narrow, the solution converges to a single stable pulse. If the width of the support increases, then several pulses emerge. Their number depends on the initial condition (Figure 3). These pulses slowly move from each other.

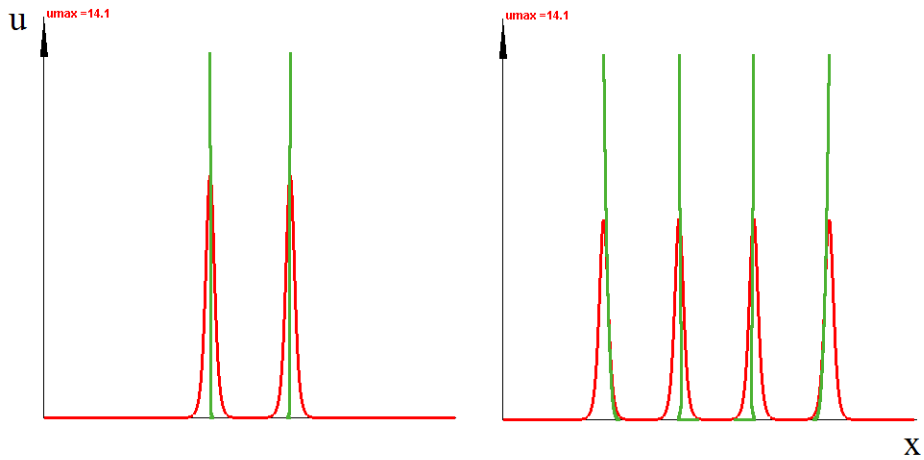

Figure 3: Multiple pulses for $N = 20$, $d = 0.2, a = 1, \sigma = 0.1$. The number of pulses depends on the width of the initial condition. Pulses slowly move from each other. Red curves show snapshots of solutions, green curves inidicate positions of maxima of solutions in time.

4. Bifurcation diagram

We summarize the result on the existence and stability of waves and pulses in the bifurcation diagram (Figure 4). It is obtained as result of numerical simulations of equation (1). We vary the value of N for all other parameters fixed. If N is sufficiently small, then there exists a stable monotone $[u_+, u_-]$- wave. The maximum of solution equals its value u_- at $-\infty$. For larger values of N, this wave still exists and it is stable but it becomes nonmonotone. We show in the diagram not the maximal value of solution in the wave but its value at $-\infty$.

At the first bifurcation point $N = N_1$, this wave becomes unstable and a periodic wave appears. It is a two stage process as described in Section 2. The diagram shows the maximal value of the periodic stationary structure which emerges behind the wave. The second bifurcation point $N = N_2$ corresponds to the transition from periodic waves to stable pulses. The average speed of the periodic wave decreases as N tends to N_2. The time interval between appearance of new localized peaks becomes longer and tends to infinity. These peaks correspond to multiple pulses for $N > N_2$. Thus we have a transition from a periodic wave to multiple pulses by means of the speed of wave propagation which becomes zero.

Let us note that both bifurcation points correspond to unusual bifurcations. In the first case, periodic waves appear due to the essential spectrum crossing the imaginary axis. In the second

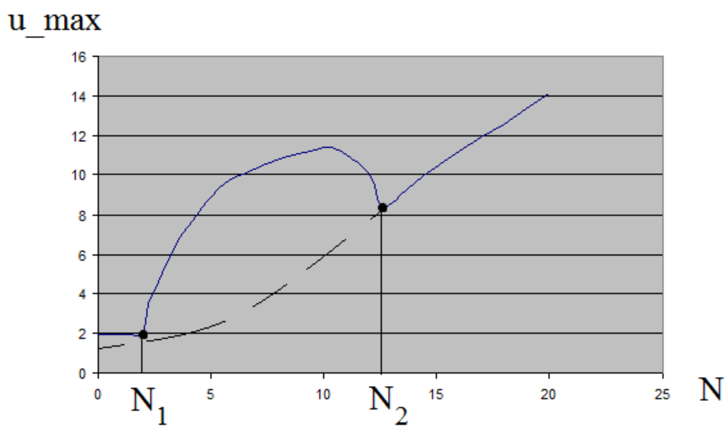

Figure 4: Bifurcation diagram: the maximum of the solution as a function of N (see the explanation in the text). Traveling wave is stable for $0 < N < N_1$, periodic travelling wave for $N_1 < N < N_2$, pulses for $N > N_2$. Solid line shows results of numerical simulations, dashed line represents qualitative behavior of solutions. The values of parameters: $d = 0.2, a = 1, \sigma = 0.1$.

case, the speed of the travelling wave approaches zero as $N \nearrow N_2$. The wave propagation can "stop" at any number of pulses.

References

[1] N. Apreutesei, A. Ducrot, V. Volpert. Travelling waves for integro-differential equations in population dynamics. Discrete Contin. Dyn. Syst., Ser. B, 11 (2009), no. 3, 541–561.

[2] N. Apreutesei, A. Ducrot, V. Volpert. Competition of species with intra-specific competition. Math. Model. Nat. Phenom., 3 (2008), 1–27.

[3] N. Bessonov, N. Reinberg, V. Volpert. Mathematics of Darwin's diagram. Math. Model. Nat. Phenom., 9 (2014), no. 3, 5–25.

[4] N.F. Britton. Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model. SIAM J. Appl. Math., 6 (1990), 1663–1688.

[5] S. Genieys, V. Volpert, P. Auger. Pattern and waves for a model in population dynamics with nonlocal consumption of resources. Math. Model. Nat. Phenom., 1 (2006), no. 1, 65–82.

[6] S.A. Gourley, M.A.J. Chaplain, F.A. Davidson. Spatio-temporal pattern formation in a nonlocal reaction-diffusion equation. Dynamical systems, 16 (2001), no. 2, 173–192.

[7] Y. Nec, M. J. Ward. The stability and slow dynamics of two-spike patterns for a class of reaction-diffusion system. Math. Model. Nat. Phenom., 8 (2013), no.5.

[8] V. Volpert. Elliptic partial differential equations. Volume 2. Reaction-diffusion equations. Birkhäuser, 2014.

[9] V. Volpert, S. Petrovskii. Reaction-diffusion waves in biology. Physics of Life Reviews, 6 (2009), 267—310.

[10] V. Volpert, V. Vougalter. Emergence and propagation of patterns in nonlocal reaction-diffusion equations arising in the theory of speciation. In: Dispersal, individual movement and spatial ecology. M. Lewis, Ph. Maini, S. Petrovskii. Editors. Springer Applied Interdisciplinary Mathematics Series. Lecture Notes in Mathematics, Volume 2071, 2013, 331-353.

[11] V. Volpert, V. Vougalter. Existence of stationary pulses for nonlocal reaction-diffusion equations. Submitted.